ALGEBRA

1. Express $x^2 - 6x + 5$ in the form $p + (x - r)^2$ and hence deduce the coordinates of the turning points of $\frac{32}{x^2 - 6x + 5}$ and state the equation of the line of symmetry.

2. If the sum of roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, show that $ab^2 + bc^2 = 2a^2c$.

3. Given that $(x+1)^2$ is a factor of the polynomial $2x^4 + 7x^3 + px^2 + qx + r$ and has a remainder of 14 when divided by $(x-1)$, find the values of $p,\ q,\ r$.

4. Given a G.P $a + b + \ldots + l$ prove that its sum is $S_n = \frac{bl - a^2}{b - a}$.

5. Given that the equations $x^2 + px + q = 0$ and $x^2 + mx + k = 0$ have a common root, show that $(q-k)^2 = (m-p)(pk - mq)$.

6. Given that one root of the equation $x^2 + px + q = 0$ is twice the other, show that $2p^2 = 9q$, hence, find the values of k, if the equation $x^2 - 2(k+2)x + (k^2 + 3k + 2) = 0$, has one root twice the other.

7. If $\log_e a^3 b = u$ and $\log_e ab^2 = v$, find a in terms of e, u and v.

8. Solve the equation: $\log_{10} e.In(x^2 + 1) - 2\log_{10} e.Inx = \log_{10} 5$

9. Prove by mathematical induction that for all positive integral values of n, $10^n + 3(4^{n+2}) + 5$ is a multiple of 9.

10i) Prove that the curve $y = \frac{4x^2 - 10x + 7}{(x-1)(x-2)}$ cannot lie in the region $-2\sqrt{3}$ and $2\sqrt{3}$.

ii) Solve the inequality: $\frac{x+1}{2x-3} \le \frac{1}{x-3}$

11. The sum of the first n terms of an A.P is $n^2 + 5n$. Find the first three terms of the series.

12. The first term of a geometric progression is A and the sum of the first three terms is $\frac{7}{4}A$.

i) Show that there are two possible progressions.

ii) Given that $A = 4$ find the next two terms of each progression.

13. Expand $\frac{1}{\sqrt{1+x}}$ up to the term in x^2 and by letting $x = \frac{1}{4}$, show that $\sqrt{5} \approx \frac{256}{115}$.

14. Express $\left(-1-\sqrt{3}i\right)^6$ in the form $x + iy$.

15. Describe the locus represented by $2|z-2i| = 3|z+1|$ and hence state the centre and radius.

16. If $\frac{(z-i)}{(z-1)}$ is purely imaginary, show that the locus of z is a circle. State the centre and radius.

17. Given $z = 1 + \cos 4\theta + i\sin 4\theta$, find the modulus and argument of z.

18. Solve for n given that ${}^nC_4 = 5 \times {}^{n-2}C_3$.

19. Find the square root of $5 + 12i$.

20. If x is sufficiently small the allow any terms in x^5 or higher powers of x to be neglected, show that $(1+x)^6\left(1-2x^3\right)^{10} \cong 1 + 6x + 15x^2 - 105x^4$.

ANALYSIS

21. Evaluate: $\int_{-1}^{-\frac{1}{2}} \frac{(4x+2)}{(x-1)^4(x+2)^4}\,dx$

22. Use small changes to show that $(16.02)^{\frac{1}{4}} \approx 2\frac{1}{1600}$

23. Use $t = \tan\frac{1}{2}x$, to find the value of $\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$

24. Sketch the curve $y=\frac{4+3x-x^2}{x-8}$, clearly find the nature of the turning points and state their asymptotes.

25. An open cylindrical container is made from a $12cm^2$ metal sheet. Show that the maximum volume of the container is $\frac{8}{\sqrt{\pi}}cm^3$.

26. The area enclosed between the parts of the curves $x^2+y^2=1$ and $4x^2+y^2=4$ for which y is positive is rotated about the $x-$ axis, find the volume of the solid generated.

27. Evaluate : $\int_1^2 (x-1)^2 \, In\, x\, dx$

28. Find Maclaurin's expansion of $y=In\frac{(2-x)^2}{(1+x)^2}$, showing the first three non zero terms, hence, find the approximate value of $2In\frac{1.99}{1.01}$ correct to 3 s.f.

29. Integrate $\int \frac{4x^2+x+1}{(x^3-1)}dx$

30. Find the area enclosed by the curves $y^2=4x$ and $x^2=4y$.

31. Evaluate: $\int_0^{1/4} \cos^{-1}2x\, dx$

32. Find the equation of the tangent at the point $(1,\ -1)$ to the curve $y=2-4x^2+x^3$. What are the coordinates of the point where the tangent meets the curve again? Find the equation of the tangent at this point.

33. If $y=\tan\left(2\tan^{-1}\frac{x}{2}\right)$, show that $\frac{dy}{dx}=\frac{4(1+y^2)}{4+x^2}$

34. Differentiate from first principles: $y=\sqrt{\cos x}$ from first principles.

35. A particle is moving in a straight line such that its distance from a fixed point O, $t\,s$ after motion begins is $s=\cos t+\cos 2t\ m$, find:

i) the time when the particle first passes through O.

ii) the velocity of the particle at this instant.

iii) the acceleration when the velocity is zero.

36. Given $\frac{dy}{dx} = 2\cos x\sqrt{y+3}$, find y in terms of x if $y\left(\frac{\pi}{2}\right) = 1$.

37. Solve the d.e $2y(x+1)\frac{dy}{dx} = 4 + y^2$, given that $y = 2$ when $x = 3$ express y in terms of x.

38. The price p of a commodity varies in such a way that the rate of change of price with respect to time t hours is proportional to the shortage $D - S$, where $D = 8 - 2p$ and $S = 2 + p$. If the price at $t = 0$ is $\$5$ and after $t = 2$ hours the price is $\$3$. Find the price of the commodity at any time and determine the price of the commodity as time tends to infinity.

39. Show that $\int_1^2 \frac{2x^3 - 1}{x^2(2x-1)}dx = \frac{3}{2} + \frac{1}{2}In\left(\frac{16}{27}\right)$

TRIGONOMETRY

40. Prove that $(\sin 2\alpha - \sin 2\beta)\tan(\alpha + \beta) = 2(\sin^2\alpha - \sin^2\beta)$.

41. Given $\sin(x+\alpha) = 2\cos(x-\alpha)$, prove that $\tan x = \frac{2 - \tan\alpha}{1 - 2\tan\alpha}$

42. Solve: $\cos(2\theta + 45^o) - \cos(2\theta - 45^o) = 1$, for $0^o \leq \theta \leq 360^o$.

43. Prove the identity: $\cos^6 x + \sin^6 x = 1 - \frac{3}{4}\sin^2 2x$.

44. The roots of the equation $ax^2 + bx + c = 0$ are $\tan\alpha$ and $\tan\beta$. Express $\text{sed}(\alpha + \beta)$ in terms of a, b, c.

45. If $a = x\cos\theta + y\sin\theta$ and $b = x\sin\theta - y\cos\theta$, prove that $\tan\theta = \frac{bx + ay}{ax - by}$.

46. Solve: $3\tan^3 x - 3\tan^2 x = \tan x - 1$ for $0^o \leq x \leq \pi$.

47. Prove that $\cos^5 x = \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{16}$.

48. Find the values of p and θ, given that $\cos(p+2)\theta + \cos p\theta = \cos\theta$ for $0^o \le \theta \le 360^o$.

49. Express $10\sin x\cos x + 12\cos 2x$ in the form $R\sin(2x+\alpha)$, hence or otherwise solve $10\sin x\cos x + 12\cos 2x + 7 = 0$ in the range $0^o \le x \le 360^o$.

50. Prove that: $\cos^2 2A + \cos^2 2B + \cos^2 2C - 1 = 2\cos 2A\cos 2B\cos 2C$.

GEOMETRY

51. The equation of a circle is given by $x^2 + y^2 + Ax + By + C = 0$, where A, B, C are constants, given that $8A = 6B,\ 6A = 4C$ and $C = 18$, find the coordinates of the centre and the radius.

52. Prove that the line $2x - 3y + 26 = 0$ is a tangent to the circle $x^2 + y^2 - 4x + 6y - 104 = 0$ and hence find the coordinates of the point of intersection.

53. Prove that the circles $x^2 + y^2 - 6x - 12y + 40 = 0$ and $x^2 + y^2 - 4y = 16$ are orthogonal.

54. A point P on the curve is given parametrically by $x = 3 - \cos\theta$ and $y = 2 + \sec\theta$.

Find the: (i) equation of the normal to the curve at the point $\theta = \dfrac{\pi}{3}$

(ii) Cartesian equation of the curve.

(b) Point $P(ap^2,\ 2ap)$ and $Q(aq^2,\ 2aq)$ lie on the parabola $y^2 = 4ax$. Find the locus of the midpoint of the chord PQ for which $pq = 2a$.

55. Find the equation of the normal to the curve $y^2 = 4bx$ at the point $P(bp^2,\ 2bp)$. Given that the normal meets the curve again at $Q(bq^2,\ 2bq)$, prove that $p^2 + pq + 2 = 0$.

56. Show that the curve $x = 5 - 6y + y^2$ represents a parabola and find the directrix and sketch.

57. The normal to the parabola $y^2 = 4ax$ at the point $P(at^2,\ 2at)$ meets the axis of the parabola at G and GP is produced, beyond P to Q so that $\overline{GP} = \overline{PQ}$. Show that the equation of locus of Q is $y^2 = 16a(x+2a)$.

58. The points $P\left(5p,\ \dfrac{5}{p}\right)$ and $Q\left(5q,\ \dfrac{5}{q}\right)$ lie on the rectangular hyperbola $xy = 25$.

i) Find the equation of the tangent at P and hence deduce the equation of the tangent at Q.

ii) The tangents at P and Q meet at point N, show that the coordinates of N are $\left(\dfrac{10pq}{p+q},\ \dfrac{10}{p+q}\right)$.

VECTORS

59i) Show that the point with position vector $\mathbf{i} - 9\mathbf{j} + \mathbf{k}$ lies on the line $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 6\mathbf{j} - \mathbf{k})$.

ii) Show that the line $\dfrac{x-2}{2} = \dfrac{2-y}{1} = \dfrac{3-z}{-3}$ is parallel to the plane $4x - y - 4z = 0$.

iii) Find the shortest distance from the point with position vector $4\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$ to the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

60i) A line passes through the mid point of $A(4,\ 3,\ 2)$ and $B(6,\ 2,\ 1)$. This line is parallel to the plane $6x + 3y + 9z = 11$. Find the equation of the line.

ii) A line passes through the point $P(1,\ -3,\ -4)$ and is perpendicular to the plane $-4x + 3y + 6z = 10$. If this line meets another plane $6x + 8y + 4z = 11$, find the point of intersection.

61. Two lines have vector equations $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (4-\beta)\mathbf{i} + (\beta+4)\mathbf{j} + (1+2\beta)\mathbf{k}$. Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.

62. A plane, P_1 passing through the point $A(1,\ 1,\ -3)$ is perpendicular to the line $\frac{x}{2}=\frac{1-y}{3}=\frac{z-3}{2}$. Another plane, P_2 contains the points with position vectors $\boldsymbol{p}=\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}$, $\boldsymbol{q}=-2\boldsymbol{i}-4\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{r}=7\boldsymbol{i}+3\boldsymbol{j}-8\boldsymbol{k}$. Find

(i) The Cartesian equations of the two planes.
(ii) The angle between the two planes.
(iii) The line of intersection of the two planes.

63. The position vectors of points A and B are **a** and **b** respectively. The point C is such that $\boldsymbol{BC}=\boldsymbol{a}$. Another point D on BC produced is such that AC intersects **OD** at T and $\boldsymbol{OT}:\boldsymbol{OD}=2:3$ with respect to the origin O.

a) Find in terms of **a** and **b**, the position vectors of (i) **BT** (ii) **TD.**

b) If $\boldsymbol{a}=15\boldsymbol{i}+\mathbf{12}\boldsymbol{j}+18\boldsymbol{k}$ and $\boldsymbol{b}=-10\boldsymbol{i}+16\boldsymbol{j}+6\boldsymbol{k}$ find to the nearest degree, the angle between **BT** and **TD.** Hence, find the area of triangle BTD.

64a) Find the equation of the plane that is perpendicular to line AB, and contains the point P , such that $AP:PB=2:1$, given that $A(1,0,-5)$ and $B(7,6,7)$.

b) The L_1 passes through the point $P(2,0,-4)$ and is parallel to the line $r=\begin{pmatrix}4\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\2\end{pmatrix}$, Determine

i) the position vector of F, the point of intersection of the plane in (a) above and the line L_1

ii) The shortest distance between $P(2,0,-4)$ and the plane in (a).

iii) $\tan\theta$ where θ is the angle between the line L_1 and the plane in (a).

65a) Find the equation of the plane containing the lines $\frac{x-2}{2}=\frac{y+3}{3}=\frac{z-1}{5}$ and $r=\begin{pmatrix}5\\1\\3\end{pmatrix}+\mu\begin{pmatrix}8\\12\\20\end{pmatrix}$.

b) Find the point of intersection of the line $\frac{x-2}{2}=\frac{y+3}{3}=\frac{z-1}{5}$ and the line given by the equation $\boldsymbol{r}=(4+2t)\boldsymbol{i}+(8-5t)\boldsymbol{j}+(7+4t)\boldsymbol{k}.$

ii) Determine the angle between the two lines in (b) above. **END**